ON THE COUETTE FLOW OF A THIRD GRADE FLUID IN A VERTICAL CHANNEL.

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Abstract

We present an analysis of the couette flow of a third Grade fluid in a horizontal channel. We show that the problem has a solution. We present an analytical solution for the Non-newtonian parameter, Electro- kinetic parameter, Brinkmann number, Electro-kinetic separation distance based on plate height, Specific internal energy.Introduction In this paper, we investigate the couette flow of a third Grade fluid in a horizotal channel.

Mebine (2007) reported the effect of thermal radiation on Magneto-Hydrodynamics (MHD) Couette flow with heat transfer between two parallel plates. In their work they investigated the problem of magnetohydrodynamics (MHD) free convection flow of an optically thin fluid bounded by two horizontal porous parallel plate Chaudhary and Jain (2007) worked on “an exact solution to the unsteady free- convection boundary –layer flow past an impulsively started vertical surface with Newtonian heating “; They submitted that an exact solution of the unsteady free – convection boundary layer flow of an incompressible fluid past an infinite vertical plate with the flow generated by Newtonian heating and impulsive motion of the plate. The use of laplace transform was employed to obtain resulting governing equations which are non – dimensional and their solutions are obtained in a closed form . A parametric study of the roles of all involved parameters is conducted and a representative set of numerical results for the velocity, temperature and skin friction is illustrated graphically. Akgul et.al.(2008) considered the electro-osmotic flow of a third grade fluid between micro-parallel plates. Approximate analytical solutions are obtained by perturbation techniques. Constant viscosity and temperature dependent viscosity cases are treated separately and they also obtained the numerical solutions, also the influence of some parameters on the velocity and temperature profiles are shown. In their work they studied the electroosmotic flow of a third grade fluid in a horizontal while in this the studied of couette flow of third grade in a ho`rizontal channel is considered.

PROBLEM FORMULATION.

The fluid is assumed to be an incompressible laminar flow. The equation of motion is given as continuity equation, momentum equation, and the energy equation as follows:

$$\nabla . v = 0 \qquad (1)$$

$$\rho \frac{dv}{dt} = \nabla \tau + \rho b \qquad (2)$$

where

$$b = \rho_e E$$

$$\rho \frac{d\zeta}{dt} = \quad \tau . grad(v) \quad \nabla q + \sigma E_X^2 \qquad (3)$$

where v is the velocity vector, ρ is the fluid density, τ is the stress tensor, b is the body force consisting of the electrical field E only with gravity not included, ρ_e is the net electric charge density, ζ is the specific internal energy, q is the heat flux vector,σ is the permittivity of electric field and$E_x^2\sigma$ term represents Joule heating, The electric field is in the y-direction only.

Method of Solution

The Poisson Boltzmann equation is related to the potential distribution within the electric double layer which can be expressed in the y-direction as follows:

$$\frac{d^2\psi^*}{dy^{*2}} = -\frac{\rho e}{\in} \quad (4)$$

Where ψ is the electrical potential, ϵ is the dielectric constant or permittivity of the fluid and the ρe is the net electric charge density. If we assume that the equilibrium Boltzmann equation is a uniform dielectric constant, the numbers of type-i ions are of the form:

$$n_i = n_{i0} \exp\left(\frac{-ze\psi^*}{k_b\theta}\right) \quad (5)$$

where $n_{i0}, z.e, k_b$ and θ are the bulk ionic concentration,valence of type –i ions elementary charge,Boltzmann's constant and absolute temperature respectively. The net electric charge density can be expressed assuming a symmetric electrolyte as follows:

$$\rho e = -2zen_0 \sinh(\psi) \quad (6)$$

Equation (6) in to (4) gives:

$$\frac{d^2\psi^*}{dy^{*2}} = \frac{2zen_0}{\in} \sinh\left(\frac{ze\psi^*}{k_b\theta^*}\right) \quad (7)$$

The Debye-Huckel linear approximation yields

$$\sinh\left(\frac{ze\psi^*}{k_b\theta^*}\right) = \left(\frac{ze\psi^*}{k_b\theta^*}\right) \quad (8)$$

Substituting equation (8) into equation (7) , we have

$$\frac{d^2\psi^*}{dy^{*2}} \cong \frac{2n_0 z^2 e^2 \psi^*}{\in k_b \theta^*} \quad (9)$$

This linear approximation is valid when the electrical potential is small compared to the thermal energy of the ions. Equation (6) can be written as

$$\frac{d^2\psi^*}{dy^{*2}} = k^2\psi^* \quad (10)$$

Where $k = ze\sqrt{\frac{2n_0}{\in k_b\theta^*}}$ is the Debye-Huckel parameter and $\frac{1}{k}$ is the Debye length. It is the characteristic thickness of the electric double layer.

The boundary conditions of equation (10) are

$$\frac{d\psi}{dy}(0) = 0,\ \psi^*(h) = \zeta^* \quad (11)$$

The non-dimensional form of equation (10) and (11) can be expressed as

$$\frac{d^2\psi}{dy^2} = k^2\psi \qquad (12)$$

$$\frac{d\psi}{dy}(0) = 0,\ \psi(h) = \zeta \qquad (13)$$

Where y is the non dimensional coordinate, ψ is the non-dimensional electric potential, k is the electro kinetic separation distance based on plate height. The non – dimensional parameters are related to the dimensional ones through the following relations:

$$y = \frac{y^*}{h}, \psi = \frac{ze\psi}{k_b\theta}, K = kh, \zeta = \frac{ze\zeta^*}{k_b\theta^*} \qquad (14)$$

The analytical solution of equation (12) subject to the boundary conditions in equation (13) gives

$$\psi = \frac{\zeta \cosh(Ky)}{\cosh K} \qquad (15)$$

Differentiating equation (15) twice, gives

$$\frac{d^2\psi}{dy^2} = \frac{\zeta K^2 \cosh(Ky)}{\cosh K} \qquad (16)$$

The viscosity is said to be a constant and using the velocity components, the momentum equation is given by:

$$\mu\frac{d^2u^*}{dy^*} + 2\beta\frac{d}{dy*}\left(\frac{du^*}{dy*}\right)^3 + \mathrm{E}_x\rho_e = \frac{\partial p^*}{\partial x^*} \qquad (17)$$

Where μ is the Kinematic viscousity, β is the material constant, E_x is the electrical field, ρ_e is the net electric charge density, ρ is density , θ_m^* is the temperature along the x axis, θ_s^* is outer temperature and g is acceleration due to gravity

$$\frac{d}{dy*}\left[(2\alpha_1 + \alpha_2)\left(\frac{du^*}{dy^*}\right)^2\right] = \frac{\partial p*}{\partial y*} \qquad (18)$$

Along the y axis

$$\frac{\partial p*}{\partial z*} = 0 \qquad (19)$$

along the z axis
If a modified pressure is defined as

$$\bar{\mathrm{p}}* = p * - \left[(2\alpha_1 + \alpha_2)\left(\frac{du^*}{dy^*}\right)^2\right] \qquad (20)$$

Equation (18) reduces to

$$\overline{p}* = p*(x) \qquad (21)$$

Equation (17) is now

$$\mu \frac{d^2 u*}{dy*} + 2\beta \frac{d}{dy*}\left(\frac{du*}{dy*}\right)^3 + E_x \rho_e$$
$$= \frac{d\overline{p}*}{dx*} \qquad (22)$$

The pressure gradient is assumed to be constant and equation (22) is

$$\mu \frac{d^2 u*}{dy*} + 2\beta \frac{d}{dy*}\left(\frac{du*}{dy*}\right)^3 + E_x \rho_e = c_0 \qquad (23)$$

Introducing the non dimensionless parameters

$$\Lambda_1 = \frac{\beta U^2}{\mu h^2}, \Lambda_2 = \frac{\varepsilon k_b \theta^* E_x}{ze\mu U}, \Lambda_3 = \frac{h^2 c_0}{\mu U},$$
$$u^* = uU, y^* = yh$$

$$\gamma_1 = \frac{\mu U^2}{K_{th}\left(\theta_m^* - \theta_s^*\right)}, \gamma_2 = \frac{\mathrm{E}_x^2 \sigma h^2}{K_{th}\left(\theta_m^* - \theta_s^*\right)},$$
$$\theta = \frac{\theta^* - \theta_s^*}{\theta_m^* - \theta_s^*} \qquad (24)$$

Where U is the reference velocity, Λ_1 is the dimensionless parameter related to the non-Newtonian behavior, Λ_2 is the dimensionless parameter for electro-kinetic effects, γ_1 is the Brinkman number, γ_2 is the dimensionless parameter related to Joule heating , θ_m^* and θ_s^* are mean and surface temperatures and c_0 the constant pressure gradient..

The non-dimensional form of equation (23) is given by

Substituting (15) into (25) to obtain

$$\frac{d^2u}{dy^2} + 6\Lambda_1\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} - \Lambda_2 \frac{\zeta K^2 \cosh(Ky)}{\cosh(K)} = \Lambda_3 \qquad (26)$$

The energy equation is given by

$$\mu\left(\frac{du^*}{dy^*}\right)^2 + 2\beta\left(\frac{du^*}{dy^*}\right)^4 + K_{th}\left(\frac{d^2\theta^*}{dy^{*2}}\right)$$
$$+ \mathrm{E}_x^2 \sigma = 0 \qquad (27)$$

Where K_{th} and σ are thermal conductivity of the fluid and the permittivity of electricity respectively, also $E_x^2\sigma$ term represents joule heating.

Therefore the non-dimensional form of equation (27) gives:

$$\frac{d^2\theta}{dy^2}+\gamma_1\left(\frac{du}{dy}\right)^2+2\Lambda_1\gamma_1\left(\frac{du}{dy}\right)^4+\gamma_2=0 \qquad (28)$$

The approximate solutions can be obtained by assuming

$$\Lambda_1=\varepsilon\lambda_1,\ \Lambda_2=\varepsilon\lambda_2, \qquad (29)$$

Where ε is the perturbation parameter in a small quantity.
To solve equation (25) and (27) let us consider the flow in a horizontal channel

Flow in horizontal channel

This is as a result of the neglect of external forces such as pressure, electroosmotic and gravitational force in the momentum equation

In solving this, a solution in terms of perturbation method is assumed in form of :

$$u=u_0+\varepsilon u_1+\varepsilon^2u_2 \qquad (30)$$

$$\theta=\theta_0+\varepsilon\theta_1+\varepsilon^2\theta_2 \qquad (31)$$

$$\frac{du}{dy}=\frac{du_0}{dy}+\varepsilon\frac{du_1}{dy}+\varepsilon^2\frac{du_2}{dy} \qquad (32)$$

s

$$\frac{d^2u}{dy^2}=\frac{d^2u_0}{dy^2}+\varepsilon\frac{d^2u_1}{dy^2}+\varepsilon^2\frac{d^2u_2}{dy^2} \qquad (34)$$

$$\frac{d^2\theta}{dy^2}=\frac{d^2\theta_0}{dy^2}+\varepsilon\frac{d^2\theta_1}{dy^2}+\varepsilon^2\frac{d^2\theta_2}{dy^2} \qquad (35)$$

$$\frac{d^2u}{dy^2}+6\varepsilon\lambda_1\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}+\varepsilon\lambda_2\frac{d^2\psi}{dy^2}=\Lambda_3 \qquad (36)$$

Substituting equation (32) and (34) into equation (26) gives $\frac{d^2u_0}{dy^2}+\varepsilon\frac{d^2u_1}{dy^2}+\varepsilon^2\frac{d^2u_2}{dy^2}+6\varepsilon\lambda_1$

$$\left(\frac{du_0}{dy}+\varepsilon\frac{du_1}{dy}+\varepsilon^2\frac{du_2}{dy}\right)^2\left(\frac{d^2u_0}{dy^2}+\varepsilon\frac{d^2u_1}{dy^2}+\varepsilon^2\frac{d^2u_2}{dy^2}\right)$$

$$-\frac{\varepsilon\lambda_2\zeta\cosh(Ky)K^2}{\cosh(K)}=\Lambda_3 \qquad (37)$$

Further simplification of equation (37)

$$\frac{d^2u_0}{dy^2}+\varepsilon\frac{d^2u_1}{dy^2}+\varepsilon^2\frac{d^2u_2}{dy^2}+6\varepsilon\lambda_1\left(\frac{du_0}{dy}\right)^2\left(\frac{d^2u_0}{dy^2}\right)$$

$$+6\varepsilon^2\lambda_1\left(\frac{du_0}{dy}\right)^2\left(\frac{d^2u_1}{dy^2}\right)+6\varepsilon^3\lambda_1\left(\frac{du_0}{dy}\right)^2\left(\frac{d^2u_2}{dy^2}\right)$$

$$+12\varepsilon^2\lambda_1\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\left(\frac{d^2u_0}{dy^2}\right)+12\varepsilon^3\lambda_1\left(\frac{du_0}{dy}\right)$$

$$\left(\frac{du_1}{dy}\right)\left(\frac{d^2u_1}{dy^2}\right)+12\varepsilon^4\lambda_1\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\left(\frac{d^2u_2}{dy^2}\right)$$

$$+12\varepsilon^3\lambda_1\left(\frac{du_0}{dy}\right)\left(\frac{du_2}{dy}\right)\left(\frac{d^2u_0}{dy^2}\right)+12\varepsilon^4\lambda_1\left(\frac{du_0}{dy}\right)$$

$$\left(\frac{du_2}{dy}\right)\left(\frac{d^2u_1}{dy^2}\right)+12\varepsilon^5\lambda_1\left(\frac{du_0}{dy}\right)\left(\frac{du_2}{dy}\right)\left(\frac{d^2u_2}{dy}\right)+6\varepsilon^3\lambda_1\left(\frac{du_1}{dy}\right)^2\left(\frac{d^2u_0}{dy}\right)+6\varepsilon^4\lambda_1\left(\frac{du_1}{dy}\right)^2\left(\frac{d^2u_1}{dy^2}\right)$$

$$+6\varepsilon^5\lambda_1\left(\frac{du_1}{dy}\right)^2\left(\frac{d^2u_2}{dy^2}\right)+12\varepsilon^6\lambda_1\left(\frac{du_1}{dy}\right)\left(\frac{du_2}{dy}\right)\left(\frac{d^2u_2}{dy^2}\right)+12\varepsilon^4\lambda_1\left(\frac{du_1}{dy}\right)\left(\frac{du_2}{dy}\right)\left(\frac{d^2u_0}{dy^2}\right)$$

$$+12\varepsilon^5\lambda_1\left(\frac{du_1}{dy}\right)\left(\frac{du_2}{dy}\right)\left(\frac{d^2u_1}{dy^2}\right)+6\varepsilon^5\lambda_1\left(\frac{du_2}{dy}\right)^2$$

$$\left(\frac{d^2u_0}{dy^2}\right)+6\varepsilon^6\lambda_1\left(\frac{du_2}{dy}\right)^2\left(\frac{d^2u_1}{dy^2}\right)+6\varepsilon^7\lambda_1$$

$$\left(\frac{du_2}{dy}\right)^2\left(\frac{d^2u_2}{dy^2}\right)\Bigg)-\frac{\varepsilon\lambda_2\zeta\cosh(Ky)K^2}{\cosh(K)}=\Lambda_3 \quad (38)$$

Now arranging in the order of ε, we have

$$6\lambda_1\left(\frac{du_2}{dy}\right)^2\left(\frac{d^2u_2}{dy^2}\right)\varepsilon^7\left(12\lambda_1\left(\frac{du_1}{dy}\right)\left(\frac{du_2}{dy}\right)\left(\frac{d^2u_2}{dy^2}\right)\right.$$

$$\left.+6\lambda_1\left(\frac{du_2}{dy}\right)^2\left(\frac{d^2u_1}{dy^2}\right)\right)\varepsilon^6+\left(6\lambda_1\left(2\left(\frac{d^2u_0}{dy^2}\right)\left(\frac{du_2}{dy}\right)\right.\right.$$

$$\left.+\left(\frac{du_1}{dy}\right)^2\right)\left(\frac{d^2u_2}{dy^2}\right)+12\lambda_1\left(\frac{du_1}{dy}\right)\left(\frac{du_2}{dy}\right)\left(\frac{d^2u_1}{dy^2}\right)$$

$$\left.+6\lambda_1\left(\frac{du_2}{dy}\right)^2\left(\frac{d^2u_2}{dy^2}\right)\right)\varepsilon^5+\left(12\lambda_1\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\left(\frac{d^2u_2}{dy^2}\right)\right.$$

$$+\left(6\lambda_1\left(2\left(\frac{du_0}{dy}\right)\left(\frac{du_2}{dy}\right)+\left(\frac{du_1}{dy}\right)^2\right)\left(\frac{d^2u_1}{dy^2}\right)\right.$$

$$\left.+12\lambda_1\left(\frac{du_1}{dy}\right)\left(\frac{du_2}{dy}\right)\left(\frac{d^2u_0}{dy^2}\right)\right)\varepsilon^4+\left(6\lambda_1\left(\frac{du_0}{dy}\right)^2\right.$$

$$\left(\frac{d^2u_2}{dy^2}\right)12\lambda_1\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\left(\frac{d^2u_1}{dy^2}\right)+6\lambda_1$$

$$\left.\left(2\left(\frac{du_0}{dy}\right)\left(\frac{du_2}{dy}\right)+\left(\frac{du_1}{dy}\right)^2\right)\left(\frac{d^2u_0}{dy^2}\right)\right)\varepsilon^3+\left(6\lambda_1\left(\frac{du_0}{dy}\right)^2\left(\frac{d^2u_1}{dy^2}\right)+12\lambda_1\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\left(\frac{d^2u_0}{dy^2}\right)\right.$$

$$\left.+\left(\frac{d^2u_2}{dy^2}\right)\right)\varepsilon^2+\left(\frac{d^2u_1}{dy^2}\right)-\frac{\lambda_2\zeta\cosh(Ky)K^2}{\cosh(K)}+6\lambda_1\left(\frac{du_0}{dy}\right)^2\left(\frac{d^2u_0}{dy^2}\right)\bigg)\varepsilon^1+\left(\frac{d^2u_0}{dy^2}\right)=\Lambda_3 \qquad (39)$$

Using the same approach applied in equation (26) for equation (28)

$$\frac{d^2\theta_0}{dy^2}+\varepsilon\left(\frac{d^2\theta_1}{dy^2}\right)+\varepsilon^2\left(\frac{d^2\theta_2}{dy^2}\right)+\gamma_1\left(\frac{du_0}{dy}\right)^2$$

$$+2\varepsilon\gamma_1\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)+2\varepsilon\lambda_1\gamma_1\left(\frac{du_0}{dy}\right)^4+8\varepsilon^2\lambda_1\gamma_1$$

$$\left(\frac{du_o}{dy}\right)^3\left(\frac{du_1}{dy}\right)+12\varepsilon^3\lambda_1\gamma_1\left(\frac{du_o}{dy}\right)^2\left(\frac{du_1}{dy}\right)^2$$

$$+8\varepsilon^4\lambda_1\gamma_1\left(\frac{du_o}{dy}\right)\left(\frac{du_1}{dy}\right)^3+2\varepsilon^5\lambda_1\gamma_1\left(\frac{du_1}{dy}\right)^4$$

$$+8\varepsilon^3\lambda_1\gamma_1\left(\frac{du_2}{dy}\right)\left(\frac{du_0}{dy}\right)^3+24\varepsilon^4\lambda_1\gamma_1\left(\frac{du_o}{dy}\right)^2$$

$$\left(\frac{du_1}{dy}\right)\left(\frac{du_2}{dy}\right)+24\varepsilon^5\lambda_1\gamma_1\left(\frac{du_o}{dy}\right)\left(\frac{du_1}{dy}\right)^2\left(\frac{du_2}{dy}\right)$$

$$+8\varepsilon^6\lambda_1\gamma_1\left(\frac{du_1}{dy}\right)^3\left(\frac{du_2}{dy}\right)+12\varepsilon^5\lambda_1\gamma_1\left(\frac{du_0}{dy}\right)^2$$

$$\left(\frac{du_2}{dy}\right)^2+24\varepsilon^6\lambda_1\gamma_1\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\left(\frac{du_2}{dy}\right)^2$$

$$+12\varepsilon^{7}\lambda_1\gamma_1\left(\frac{du_1}{dy}\right)^2\left(\frac{du_2}{dy}\right)^2+8\varepsilon^{7}\lambda_1\gamma_1\left(\frac{du_0}{dy}\right)$$

$$\left(\frac{du_2}{dy}\right)^3+8\varepsilon^{8}\lambda_1\gamma_1\left(\frac{du_1}{dy}\right)\left(\frac{du_2}{dy}\right)^3$$

$$+2\varepsilon^{9}\lambda_1\gamma_1\left(\frac{du_2}{dy}\right)^4+\gamma_2=0 \qquad (40)$$

Arranging in order of ε

$$2\lambda_1\gamma_1\left(\frac{du_2}{dy}\right)^4\varepsilon^9+8\lambda_1\gamma_1\left(\frac{du_1}{dy}\right)\left(\frac{du_2}{dy}\right)^3\varepsilon^8$$

$$+2\lambda_1\gamma_1\left(2\left(2\left(\frac{du_0}{dy}\right)\right)\left(\frac{du_2}{dy}\right)+\left(\frac{du_1}{dy}\right)^2\right)\left(\frac{du_2}{dy}\right)^2$$

$$+4\left(\frac{du_1}{dy}\right)^2\left(\frac{du_2}{dy}\right)^2\Bigg)\varepsilon^7+2\lambda_1\gamma_1\left(4\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\right.$$

$$\left(\frac{du_2}{dy}\right)^2+4\left(2\left(\frac{du_0}{dy}\right)\left(\frac{du_2}{dy}\right)+\left(\frac{du_1}{dy}\right)^2\right)\left(\frac{du_1}{dy}\right)\left(\frac{du_2}{dy}\right)\Bigg)\varepsilon^6$$

$$+2\lambda_1\gamma_1\left(2\left(\frac{du_0}{dy}\right)^2\left(\frac{du_2}{dy}\right)^2+8\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)^2\left(\frac{du_2}{dy}\right)\right.$$

$$+\left(2\left(\frac{du_0}{dy}\right)\left(\frac{du_2}{dy}\right)+\left(\frac{du_1}{dy}\right)^2\right)^2\Bigg)\varepsilon^5+2\lambda_1\gamma_1\left(2\left(\frac{du_0}{dy}\right)^2\right.$$

$$\left(\frac{du_2}{dy}\right)^2\left(\frac{du_1}{dy}\right)\left(\frac{du_2}{dy}\right)+4\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\left(2\left(\frac{du_0}{dy}\right)\left(\frac{du_2}{dy}\right)\right.$$

$$\left(2\left(\frac{du_0}{dy}\right)\left(\frac{du_2}{dy}\right)^3+\left(\frac{du_1}{dy}\right)^2\right)\Bigg)\Bigg)\varepsilon^4+\left(2\gamma_1\left(\frac{du_1}{dy}\right)\left(\frac{du_2}{dy}\right)\right.$$

$$+2\lambda_1\gamma_1\left(2\left(\frac{du_0}{dy}\right)^2\left(2\left(\frac{du_0}{dy}\right)\left(\frac{du_2}{dy}\right)^3+\left(\frac{du_1}{dy}\right)^2\right)+4\left(\frac{du_0}{dy}\right)^2\right.$$

$$\left(\frac{du_1}{dy}\right)^2\Bigg)\Bigg)\varepsilon^3+\left(\frac{d^2\theta_2}{dy^2}+\gamma_1\left(2\left(\frac{du_0}{dy}\right)\left(\frac{du_2}{dy}\right)+\left(\frac{du_1}{dy}\right)^2\right)\right.$$

$$+8\lambda_1\gamma_1\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\Bigg)\varepsilon^2+\left(\frac{d^2\theta_1}{dy^2}+2\gamma_1\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\right.$$

$$\left.+2\lambda_1\gamma_1\left(\frac{du_0}{dy}\right)^4\right)\varepsilon+\frac{d^2\theta_0}{dy^2}+\gamma_1\left(\frac{du_0}{dy}\right)^2+\gamma_2=0 \quad (41)$$

Considering the zeroth order of ε in equation (41)

$$o\left(\varepsilon^0\right):\frac{d^2u_0}{dy^2}=\Lambda_3$$

Integrating twice

$$u_0=\frac{\Lambda_3 y^2}{2}+c_1 y+c_2 \quad (42)$$

Applying the boundary conditions

$u_0=0, u_0(1)=1$;

$c_2=0; \quad c_1=1-\frac{\Lambda_3}{2}$ Now, substituting the value of c_1 and c_2 in equation (42)

$$u_0=\frac{\Lambda_3 y^2}{2}+\left(1-\frac{\Lambda_3}{2}\right)y \quad (43)$$

Considering the first order of ε in equation (41)

First Order: $O(\varepsilon^1)$

$$\frac{d^2u_1}{dy^2}=-6\lambda_1\left(\frac{du_0}{dy}\right)^2\left(\frac{d^2u_0}{dy^2}\right)+\lambda_2\frac{K^2\zeta\cosh(Ky)}{\cosh K} \quad (44)$$

Differentiating equation (43) and substituting into equation (44)

$$\frac{d^2u_1}{dy^2}=-6\lambda_1\left(\Lambda_3 y-1+\frac{\Lambda_3}{2}\right)^2(\Lambda_3)+\lambda_2\frac{K^2\zeta\cosh(Ky)}{\cosh K}$$

$$\frac{d^2u_1}{dy^2}=-6\lambda_1\left(\Lambda_3^2 y^2+2\Lambda_3 y+1+\Lambda_3^2 y-\Lambda_3+\frac{\Lambda_3^2}{4}\right)(\Lambda_3)+\lambda_2\frac{K^2\zeta\cosh(Ky)}{\cosh K}$$

$$\frac{d^2u_1}{dy^2}=-6\lambda_1\Lambda_3^3 y^2-12\lambda_1\Lambda_3^2 y-6\lambda_1\Lambda_3-6\lambda_1\Lambda_3^3 y$$

$$+6\lambda_1\Lambda_3^2-\frac{3}{2}\lambda_1\Lambda^3{}_3+\frac{\lambda_2 K^2\zeta\cosh(Ky)}{\cosh K} \quad (45)$$

Applying the boundary conditions

$$u_1(0)=0,\ u_1(1)=1 \tag{46}$$

$$c_6 = -\frac{\lambda_2\zeta}{\cosh k}$$

$$1 = -\frac{1}{2}\lambda_1\Lambda_3^3 - 2\lambda_1\Lambda_3^2 + \lambda_1\Lambda_3^3 - \frac{3}{4}\lambda_1\Lambda_3 + \frac{3}{2}\lambda_1\Lambda_3^2$$
$$-\frac{3}{4}\lambda_1\Lambda_3^3 + \frac{\lambda_2\zeta\cosh(K)}{\cosh \mathrm{K}} + c_5 + c_6$$

$$c_5 = -\frac{1}{4}\left(\frac{-\lambda_1\Lambda_3^3\cosh(K) - 2\lambda_1\Lambda_3^2\cosh(K) + 4\lambda_2\zeta\cosh(K)}{\cosh(K)}\right.$$
$$\left.\frac{-3\lambda_1\Lambda_3\cosh(K) - 4\Lambda_2\zeta - 4\cosh(K)}{\cosh(K)}\right)$$

Substituting c_5 and c_6 gives the solution for u_1

Considering the second order of ε in equation (39)

Second Order: $0(\varepsilon^2)=$

$$\frac{d^2u_2}{dy^2} + 6\lambda_1\left(\frac{du_0}{dy}\right)^2\left(\frac{d^2u_1}{dy^2}\right) + 12\lambda_1\frac{du_0}{dy}\frac{du_1}{dy}\frac{d^2u_0}{dy^2} = 0 \tag{47}$$

Differential equation (43) and (47) and substituting into equation (48)

$$\frac{d^2u_2}{dy^2} = -6\lambda_1\left(\Lambda_3 y + 1 - \frac{\Lambda_3}{2}\right)^2\left(\frac{1}{2}\frac{1}{\cosh(K)}\right.$$

$$\left(-12\lambda_1\Lambda_3^3\cosh(K)y^2 - 24\lambda_1\Lambda_3^2\cosh(K)y + 12\lambda_1\Lambda_3^3\right.$$

$$\cosh(K)y - 12\lambda_1\Lambda_3\cosh(K) + 12\lambda_1\Lambda_3^2\cosh(K)$$

$$\left.\left.-3\lambda_1\Lambda_3^3\cosh(K) + 2\lambda_2\zeta\cosh(Ky)\right)K^2\right) - 12\lambda_1$$

$$\left(\Lambda_3 y + 1 - \frac{\Lambda_3}{2}\right)\left(\frac{1}{2}\frac{1}{\cosh(k)}\left(-12\lambda_1\Lambda_3^3\cosh(K)y^2\right.\right.$$

$$-24\lambda_1\Lambda_3^2\cosh(K)y + 12\lambda_1\Lambda_3^3\cosh(K)y - 12\lambda_1\Lambda_3$$
$$\cosh(K) + 12\lambda_1\Lambda_3^2\cosh(K) - 3\lambda_1\Lambda_3^3\cosh(K) + 2\lambda_2\zeta$$

$$\cosh(Ky)K^2 - \frac{1}{4}\frac{1}{\cosh(K)}(-4\lambda_2\zeta - \lambda_1\Lambda_3^3\cosh(K)$$
$$+4\lambda_1\Lambda_3^2\cosh(K) - 12\lambda_1\Lambda_3\cosh(K) + 4\lambda_2\zeta\cosh(K)y)$$

$$\left(\frac{1}{2}\frac{1}{\cosh(K)}\left(-12\lambda_1\Lambda_3^3\cosh(K)y^2 - 24\lambda_1\Lambda_3^2\cosh(K)y + \right.\right.$$
$$12\lambda_1\Lambda_3^3\cosh(K)y - 12\lambda_1\Lambda_3\cosh(K) + 12\lambda_1\Lambda_3^2\cosh(K)$$
$$-3\lambda_1\Lambda_3^3\cosh(K) + 2\lambda_2\zeta\cosh(Ky)K^2) \qquad (48)$$

Integrating equation (49) twice and applying the boundary conditions

zeroth Order: $0(\varepsilon^0)$

$$O(\varepsilon^0) = \frac{d^2\theta_0}{dy^2} + \gamma_1\left(\frac{du_0}{dy}\right)^2 + \gamma_2 \qquad (49)$$

Differentiating equation (43)and substituting into equation (49)

$$\frac{d^2\theta_0}{dy^2} = -\gamma_1\left(\Lambda_3 y + 1 - \frac{\Lambda_3}{2}\right)^2 - \gamma_2$$
$$\frac{d^2\theta_0}{dy^2} = -\gamma_1\left(\Lambda_3^2 y^2 + 2\Lambda_3 y - \Lambda_3^2 y + 1 - \Lambda_3 + \frac{\Lambda_3^2}{4}\right) - \gamma_2 \qquad (50)$$

Integrating equation (50) and applying the boundary conditions $\theta_0(0)=0,\ \theta_0(1)=0$ gives

$$c_1 = \frac{1}{12}\left(\frac{\gamma_1\Lambda_3^2 + 4\gamma_1 + 6\gamma_2\Lambda_3}{\Lambda_3}\right)$$
$$c_2 = \frac{1}{192}\left(\frac{\gamma_1\left(16 - 32\Lambda_3 + 24\Lambda_3^2 - 8\Lambda^3{}_3 + \Lambda_4^3\right)}{\Lambda^2{}_3}\right)$$

So that,

$$\theta_0 = -\frac{1}{192}\frac{\gamma_1(2\Lambda_3 y - \Lambda_3 + 2)^4}{\Lambda_3^2} - \frac{1}{2}\gamma_2 y^2$$
$$+\frac{1}{12}\frac{\left(\gamma_1\Lambda_3^2 + 12\Lambda_3 + 6\Lambda_3\gamma_2 + 4\gamma_1\right)y}{\Lambda_3}$$
$$+\frac{1}{192}\frac{\gamma_1\left(\Lambda_3^4 - 8\Lambda_3^3 + 24\Lambda_3^2 - 32\Lambda_3 + 16\right)}{\Lambda_3^2}$$

First Order: $0(\varepsilon^1)$

$$0(\varepsilon^1): \frac{d^2\theta_1}{dy^2} = -2\gamma_1\left(\frac{du_0}{dy}\right)^4 - 2\gamma_1\frac{du_0}{dy}\frac{du_1}{dy} \quad (51)$$

Differentiating equation (42.) and (47) and substituting into equation (51)

$$\frac{d^2\theta_1}{dy^2} = -2\lambda_1\gamma_1\left(\Lambda_3 y + 1 - \frac{\Lambda_3}{2}\right)^4 - 2\gamma_1\left(\Lambda_3 y + 1 - \frac{\Lambda_3}{2}\right)$$
$$\left(\frac{1}{2}\frac{1}{\cosh(K)}\left(-4\lambda_1\Lambda_3^3\cosh(K)y - 12\lambda_1\Lambda_3^2\cosh(K)y^2 + \right.\right.$$

$$6\lambda_1\Lambda_3^3\cosh(K)y^2 - 12\lambda_1\Lambda_3\cosh(K)y + 12\lambda_1\Lambda_3^2\cosh(K)y^2$$
$$-3\lambda_1\Lambda_3^3\cosh(K)y + 2\lambda_1\zeta\sinh(Ky)y)) - \frac{1}{4}\frac{1}{\cosh(K)}$$

$$(-4\lambda_2\zeta - \lambda_1\Lambda_3^3\cosh(K) + 4\lambda_1\Lambda_3^2\cosh(K) -$$
$$12\lambda_1\Lambda_3\cosh(K) + 4\lambda_2\zeta\cosh(K)) \quad (54)$$

Integrating equation (54) twice and substituting the boundary conditions we have θ_1

Second Order: $0(\varepsilon^2)$

$$0(\varepsilon^2): \frac{d^2\theta_2}{dy^2} = -\gamma_1\left(2\frac{du_0}{dy}\left(\frac{du_2}{dy}\right) + \left(\frac{du_1}{dy}\right)^2\right)$$

$$-8\lambda_1\gamma_1\left(\frac{du_0}{dy}\right)^3\frac{du_1}{dy} \quad (56)$$

Differentiating equation (42) and (47) and substituting into equation (56)

$$\frac{d^2\theta_2}{dy^2} = -2\gamma_1\left(\Lambda_3 y + 1 - \frac{\Lambda_3}{2}\right)\left(\frac{1}{8}\frac{1}{\cosh(K)K^2}\right.$$

$$\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2$$

$$\theta = -\frac{1}{192}\frac{\gamma_1(2\Lambda_3 y - \Lambda_3 + 2)^4}{\Lambda_3^2} - \frac{1}{2}\gamma_2 y^2 + \frac{1}{12}$$
$$\frac{(\gamma_1\Lambda_3^2 + 12\Lambda_3 + 6\Lambda_3\gamma_2 + 4\gamma_1)y}{\Lambda_3} + \frac{1}{192}$$
$$\frac{\gamma_1(\Lambda_3^4 - 8\Lambda_3^3 + 24\Lambda_3^2 - 32\Lambda_3 + 16)}{\Lambda_3^2} + \frac{1}{K^2\cosh(K)}$$

$$
\begin{aligned}
&(0.00000416666667\gamma_1(96\lambda_1\Lambda_3^3\cosh(K)K^2y^5 \\
&+240\lambda_1\Lambda_3^2\cosh(K)K^2y^4+160\lambda_1\Lambda_3\cosh(K)K^2y^3 \\
&+16\Lambda_3^4K^2\lambda_1\cosh(K)y^6+240\lambda_2\zeta\cosh(K)K^2y^2 \\
&-80\Lambda_3\lambda_2\zeta K^2y^3+80\Lambda_3\lambda_2\zeta K^2\cosh(K)y^3-96080\Lambda_3\lambda_2\zeta \\
&-480\Lambda_3y\lambda_2\zeta K\sinh(Ky)-120\Lambda_3\lambda_2\zeta\cosh(K)y^2K^2 \\
&-480\lambda_2\zeta K\sinh(Ky)+120\Lambda_3\lambda_2\zeta y^2K^2-60\lambda_1\Lambda_3^3\cosh(K)y^2K^2 \\
&+15\Lambda_3^4y^2\lambda_1\cosh(K)K^2-40\Lambda_3^4y^3\lambda_1\cosh(K)K^2 \\
&+60\lambda_1\Lambda^4{}_3\cosh(K)y^4K^2+200\lambda_1\Lambda_3^3\cosh(K)y^3K^2 \\
&+240\lambda_1\Lambda_3^2\cosh(K)y^2K^2-240\lambda_1\Lambda_3\cosh(K)y^2K^2 \\
&-48\lambda_1\Lambda_3^4\cosh(K)y^5K^2-240\lambda_1\Lambda_3^3\cosh(K)y^4K^2 \\
&-480\lambda_1\Lambda_3^2\cosh(K)y^3K^2+240\Lambda_3\lambda_2\zeta K\sinh(Ky) \\
&-240\Lambda_3\lambda_2\zeta K^2y-240\lambda_2\zeta K^2y^2+480\lambda_2\zeta K^2y-240\lambda_1 \\
&\cosh(K)K^2y^2+960\Lambda_3\lambda_2\zeta\cosh(Ky)))+\frac{1}{K^2\cosh(K)} \\
&(0.00000416666667\gamma_1(960\Lambda_3\lambda_2\zeta e^K-240\lambda_2\zeta\cosh(K)K^2e^K \\
&-480\Lambda_3\lambda_2\zeta-240\lambda_2\zeta K^2e^K+240\lambda_2\zeta Ke^{2K}-3\lambda_1\Lambda_3^4 \\
&\cosh(K)K^2e^K+240\zeta K\lambda_2e^{2K}-3\Lambda^4{}_3K^2\lambda_1\cosh(K)e^K \\
&-240\zeta K\lambda_2+4\lambda_1\Lambda_3^3\cosh(K)K^2e^K+80\lambda_1\Lambda_3\cosh(K) \\
&K^2e^K+240\lambda_1\cosh(K)K^2e^K+200\Lambda_3\lambda_2\zeta K^2e^K \\
&+40\Lambda_3\lambda_2\zeta\cosh(K)K^2e^K-120\Lambda_3\lambda_2\zeta K-480\Lambda_3\lambda_2 \\
&\zeta e^{2K}+120\Lambda_3\lambda_2\zeta e^{2K})e^{-K}y)+\frac{1}{K^4\cosh(K)^2} \\
&(2.9676190476X10^{-10}\gamma_1(13440\lambda_1\cos(K)\Lambda_3^3y^3\lambda_2 \\
&\zeta K^2+645120\lambda_2\zeta\lambda_1\Lambda_3^3\cosh(K)-3360K^3\lambda_2\zeta\lambda_1\Lambda_3^3 \\
&\cosh(K)\sinh(Ky)+16800K^3\lambda_2\zeta\lambda_1\Lambda_3^2\cosh(K)\sinh(Ky) \\
&-3360\lambda_1\cosh(K)\Lambda_3^2y^3\lambda_2\zeta K^4e^K+3360\lambda_1\cosh(K)\Lambda_3^3\lambda_2\zeta \\
&K^4y+1260\lambda_1^2\cosh(K)^2\Lambda_3^6y^3K^4-3360\lambda_1\cosh(K)\Lambda_3^3 \\
&y^3e^{-K}\lambda_2\zeta K^3-13440\lambda_1^2\cosh(K)^2\Lambda_3y^3K^4-80640K^3 \\
&\lambda_2\zeta\lambda_1\Lambda_3^2\cosh(K)y\sinh(Ky)+2100\lambda_1^2\cosh(K)^2\Lambda_3^5y^2 \\
&K^4-840\lambda_1\cosh(K)\Lambda_3^3y^3e^{-K}\lambda_2\zeta K^4-5040\lambda_1\cosh(K) \\
&\Lambda_3y^2e^{-K}\lambda_2\zeta K^4+13440\lambda_1\cosh(K)^2\Lambda_3^2\lambda_2\zeta y^3K^4 \\
&+10080\lambda_1\cosh(K)^2\Lambda_3\lambda_2\zeta y^2K^4-20160\lambda_1\cosh(K)\Lambda_3^2 \\
&\lambda_2\zeta y^3K^4-40320\lambda_1\cosh(K)\Lambda_3\lambda_2\zeta K^4y^2-40320K^3\lambda_2 \\
&\zeta\lambda_1\Lambda_3\cosh(K)\sinh(Ky)+161280\lambda_1\cosh(K)\Lambda_3^2\lambda_2\zeta K^2
\end{aligned}
$$

$\cosh(Ky) + 20160\Lambda_3 y^2 \lambda_1 \lambda_2 \zeta e^K \cosh(K) K^3 + 6720 K^4$
$\lambda_2^2 \zeta^2 y - 1344\lambda_1 \cosh(K)^2 \Lambda_3^3 y^5 \lambda_2 \zeta K^4 - 6720\lambda_1 \cosh(K)^2$
$\Lambda_3^2 \lambda_2 \zeta y^4 K^4 + 80640\lambda_1 \cosh(K) \Lambda_3^2 \lambda_2 \zeta K^3 y^2 \sinh(Ky)$
$+1260\lambda_1 \cosh(K) \Lambda_3^3 y^2 e^{-K} \lambda_2 \zeta K^4 - 10080\lambda_1 \cosh(K) e^{-K}$
$\lambda_2 \zeta K^4 y^2 + 13440\lambda_1 \cosh(K) \lambda_2 \zeta K^3 \sinh(Ky) - 840\lambda_1$
$\cosh(K) \Lambda_3^3 y^3 \lambda_2 \zeta K^4 e^K + 840 K^4 \lambda^2{}_2 \zeta^2 - 1680 K^4$
$\lambda^2{}_2 \zeta^2 \cosh(K)^2 y^2 - 36960\lambda^2{}_1 \cosh(K)^2 \Lambda_3^2 y^4 K^4$
$-37632\lambda^2{}_1 \cosh(K)^2 \Lambda_3^3 y^5 K^4 - 20160\lambda^2{}_1 \cosh(K)^2$
$\Lambda_3^4 y^6 K^4 - 20160 K^3 \lambda_2 \zeta \lambda_1 \Lambda_3^3 \cosh(K) y \sinh(Ky)$
$-5040\lambda_1 \cosh(K) \Lambda_3^3 y^2 \lambda_2 \zeta K^3 e^K + 6720\lambda_1 \cosh(K)$

$\Lambda_3^2 \lambda_2 \zeta y^4 K^4 + 13440\lambda_1 \cosh(K) \Lambda_3 \lambda_2 \zeta K^4 y^3 - 16800\lambda_1$
$\cosh(K) \Lambda_3^2 \lambda_2 \zeta K^4 y + 840\lambda_1 \cosh(K)^2 \Lambda_3^3 y^2 \lambda_2 \zeta K^4$
$-6720 K^4 \lambda_2^2 \cosh(K) y + 2880\lambda_1^2 \cosh(K)^2 \Lambda_3^6 y^7 K^4$
$-3360\lambda_1 \cosh(K) \Lambda_3 y^3 e^{-K} \lambda_2 \zeta K^4 + 10080\lambda_1 \cosh(K)$
$+2520\lambda_1 \cosh(K) \Lambda_3^2 y^2 \lambda_2 \zeta K^4 e^K + 3360\lambda_1 \cosh(K)$
$\Lambda_3^3 y^3 \lambda_2 \zeta K^3 e^K + 80640 K^3 \lambda_2 \zeta \lambda_1 \cosh(K) y \sinh(Ky)$
$+161280\lambda_1 \cosh(K) \Lambda_3^3 \lambda_2 \zeta K^2 y \cosh(Ky) + 483840\lambda_2 \zeta \lambda_1$
$\Lambda_3^3 \cosh(K) K y \sinh(Ky) - 161280 K^2 \lambda_2 \zeta \lambda_1 \Lambda_3 \cosh(K)$
$\cosh(Ky) - 241920\lambda_2 \zeta \lambda_1 \Lambda_3^3 \cosh(K) \sinh(Ky) K$
$+6720\lambda_1 \cosh(K) K^4 \lambda_2 \zeta y^2 - 29568\lambda_1^2 \cosh(K)^2 \Lambda_3^5 y^5 K^4$
$-3360\lambda_1 \cosh(K) \Lambda_3^3 y^4 \lambda_2 \zeta K^4 + 3360\lambda_1 \cosh(K)^2 \Lambda_3^3 y^4 \lambda_2 \zeta K^4$
$+26880\lambda_2 \zeta \lambda_1 \Lambda_3^3 \cosh(K) K^4 y^3 \sinh(Ky) - 20160\lambda_1 \cosh(K)$
$\Lambda_3 y^2 e^{-K} \lambda_2 \zeta K^3 - 6720\lambda_1 \cosh(K) \Lambda_3^2 y^3 e^{-K} \lambda_2 \zeta K^4 - 10360\lambda_1^2$
$\cosh(K)^2 \Lambda_3^5 y^3 K^4 + 20160\lambda_1^2 \cosh(K)^2 \Lambda_3^5 y^6 K^4 - 6720\lambda_1$
$\cosh(K)^2 \Lambda_3 \lambda_2 \zeta y^3 K^4 - 3360\Lambda_3 y^3 \lambda_1 \lambda_2 \zeta e^K \cosh(K) K^4$
$\cosh(K)^2 \Lambda_3 \lambda_2 \zeta y^3 K^4 - 40320\lambda_1 \cosh(K) \Lambda_3^3 \lambda_2 \zeta K^3 y^2 \sinh(Ky)$
$+23520\lambda_1^2 \cosh(K)^2 \Lambda_3^5 y^4 K^4 + 6720\lambda_1 \cosh(K) \Lambda_3^2 y^2 \lambda_2 \zeta K^3 e^K$
$+483840\lambda_2 \zeta \lambda_1 \Lambda_3^2 \cosh(K) \sinh(Ky) K - 3150\lambda_1^2 \cosh(K)^2$
$\Lambda_3^6 y^4 K^4 - 5760\lambda_1^2 \cosh(K)^2 \Lambda_3^5 y^7 K^4 + 6720 K^3 \lambda_2^2 \zeta^2 \cosh(K)$
$\sinh(Ky) + 10080\lambda_1 \cosh(K) \Lambda_3^3 y^2 \lambda_2 \zeta K^2 e^K + 1260\lambda_1 \cosh(K)$
$\Lambda_3^3 y^2 \lambda_2 \zeta K^4 e^K + 73920\lambda_1^2 \cosh(K)^2 \Lambda_3^2 y^3 K^4 + 94080\lambda_1^2$
$\cosh(K)^2 \Lambda_3^3 y^4 K^4 - 20160\lambda_1 \cosh(K) \Lambda_3^2 y^2 e^{-K} \lambda_2 \zeta K^2$
$-6720\lambda_1 \cosh(K) \Lambda_3^3 y^3 e^{-K} \lambda_2 \zeta K^2 - 5040\Lambda_3 y^2 \lambda_1 \lambda_2 \zeta e^K$
$\cosh(K) K^4 + 40320 K^2 \lambda_2 \zeta \lambda_1 \Lambda_3^3 \cosh(K) - 161280 K^2 \lambda_2 \zeta \lambda_1$

$$\Lambda_3^2\cosh(K)+161280K^2\lambda_2\zeta\lambda_1\Lambda_3\cosh(K)-80640\lambda_1^2\cosh(K)^2$$
$$\Lambda_3^3y^3K^4-72240\lambda_1^2\cosh(K)^2\Lambda_3^4y^4K^4+13440\lambda_1\cosh(K)^2$$
$$K^4\lambda_2\zeta y^2-322560\lambda_1\cosh(K)\Lambda_3^2\lambda_2\zeta K^2y\cosh(Ky)-6720K^3$$
$$\lambda_2^2\zeta^2\sinh(Ky)+20160\lambda_1^2\cosh(K)^2\Lambda_3y^2K^4-20160\lambda_1$$
$$\cosh(K)\Lambda_3^3y^2\lambda_2\zeta K^2-12600\lambda_1^2\cosh(K)^2\Lambda_3^4y^2K^4+26880\lambda_1^2$$
$$\cosh(K)^2\Lambda_3^3y^2K^4-840K^4\lambda_2^2\zeta^2\cosh(Ky)^2-10080\lambda_1\lambda_2\zeta e^K$$
$$\cosh(K)K^4y^2+1344\lambda_1\cosh(K)\Lambda_3^3y^5\lambda_2\zeta K^4+840K^6\lambda_2^2\zeta^2y^2$$
$$-645120\lambda_2\zeta\lambda_1\Lambda_3^3\cosh(K)\cosh(Ky)-1680K^4\lambda_2^2\zeta^2y^2$$
$$-161280\lambda_2\zeta\lambda_1\Lambda_3^3\cosh(K)K^2y^2\cosh(Ky)+43680\lambda_1^2\cosh(K)^2$$
$$\Lambda_3^4y^3K^4-31920\lambda_1^2\cosh(K)^2\Lambda_3^2y^2K^4+3360K^4\lambda_2^2\zeta^2$$
$$\cosh(K)y^2-720\lambda_1^2\cosh(K)^2\Lambda_3^6y^8K^4-20160\Lambda_3^2y^2\lambda_1\lambda_2\zeta e^K$$
$$\cosh(K)K^2-3360\lambda_1\cosh(K)\Lambda_3^2y^3e^{-K}\lambda_2\zeta K^4y^2$$
$$+80640\lambda_1\Lambda_3^3\cosh(K)y\lambda_2\zeta K^2-161280\lambda_2\zeta\lambda_1\Lambda_3^2\cosh(K)yK^2$$
$$+2520\lambda_1\cosh(K)\Lambda_3^2y^2e^{-K}\lambda_2\zeta K^4+5040\lambda_1^2\cosh(K)^2\Lambda_3^6y^5K^4$$
$$-5040\lambda_1^2\cosh(K)^2\Lambda_3^6y^6K^4-13440\lambda_1\cosh(K)K^4$$
$$\lambda_2\zeta y+4480\lambda_1\cosh(K)\Lambda_3^3y^3\lambda_2\zeta K^4-3360\lambda_1\cosh(K)$$
$$\Lambda_3^3y^2\lambda_2\zeta K^4-2800\lambda_1\cosh(K)^2\Lambda_3^3y^3\lambda_2\zeta K^4-6720\lambda_1\cosh(K)^2$$
$$\Lambda_3^2\lambda_2\zeta y^2K^4+21840\lambda_1\cosh(K)^2\Lambda_3^2\lambda_2\zeta y^2K^4-40320\lambda_1\cosh(K)$$
$$\Lambda_3^3\lambda_2\zeta K^2\cosh(Ky)+60480\lambda_1^2\cosh(K)^2\Lambda_3^4y^5K^4))$$
$$-\frac{1}{\cosh(K)^2K^4}(2.976190476\times10^{-10}\gamma_1-8400\Lambda_3\lambda_1\lambda_2$$
$$\zeta e^3K\cosh(K)K^4-10080\lambda_1\lambda_2\zeta e^K\cosh(K)K^4-10080$$
$$\lambda_1\cosh(K)\Lambda_3^2\lambda_2\zeta K^2e^{3K}-80640\lambda_1\cosh(K)\Lambda_3\lambda_2\zeta K^2e^K$$
$$-16800\lambda_1\cosh(K)\Lambda_3^3\lambda_2\zeta K^2e^K-10080\lambda_1\cosh(K)\Lambda_3^2$$
$$\lambda_2\zeta K^2e^K-120960\lambda_1\Lambda_3^3\lambda_2\zeta e^K\cosh(K)K-8400\lambda_1$$
$$\cosh(K)\Lambda_3^2\lambda_2\zeta K^4e^{2K}+13440\lambda_1\cosh(K)\Lambda_3\lambda_2\zeta K^4e^{2K}$$
$$+2688\lambda_1^2\cosh(K)^2\Lambda_3^3K^4e^{2K}+92\lambda_1^2\cosh(K)^2\Lambda_3^5K^4e^{2K}$$
$$-3360K^4\lambda_2^2\zeta^2\cosh(K)e^{2K}+3360K^3\lambda_2^2\zeta^2\cosh(K)e^{3K}$$
$$-840\lambda_1^2\cosh(K)^2\Lambda_3^4K^4e^{2K}-210K^4\lambda_2^2\zeta^2-3360K^3\lambda_2^2$$
$$\zeta^2\cosh(K)e^K+15120\lambda_1\cosh(K)\Lambda_3^2\lambda_2\zeta K^3e^{3K}$$

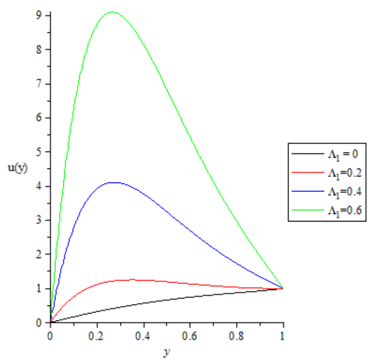


Figure 1: Dimensionless velocity profile for different NonNewtonian parameter $\Lambda_1(\zeta = 1, K = 5, \Lambda_2 = 0.2, \Lambda_3 = -2)$.

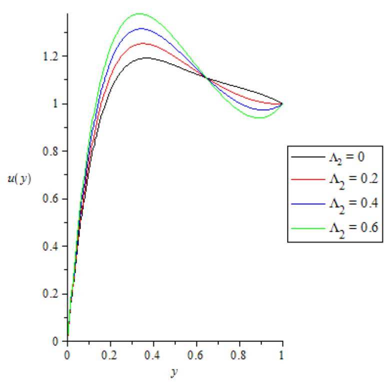


Figure 2: Dimensionless Velocity profile for different Electro-kinetic parameter when $(\zeta = 1, K = 5.\Lambda_1 = 0.2, \Lambda_3 = -2)$

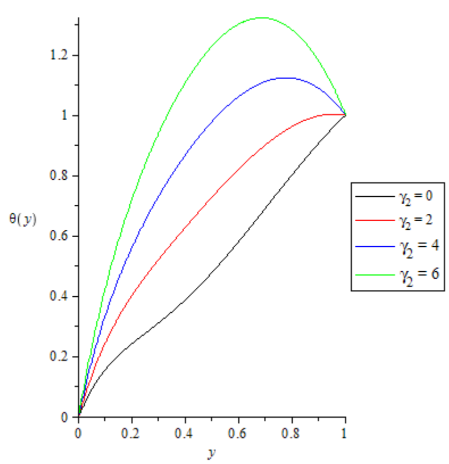


Figure 3: Dimensionless Temperature distribution for different Joule Heating parameter γ_2 when $(\zeta = 1, K = 5.\Lambda_1 = 0.1, \Lambda_2 = 0.2, \Lambda_3 = -2, \gamma_1 = 2)$

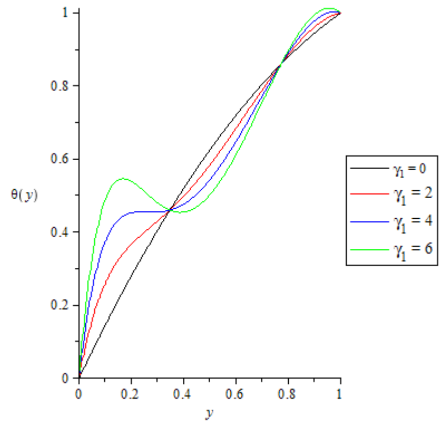


Figure.4: Dimensionless Temperature distribution for different Brinkman number γ_1 when $(\zeta = 1, K = 5.\Lambda_1 = 0.1, \Lambda_2 = 0.2, \Lambda_3 = -2, \gamma_2 = 2)$

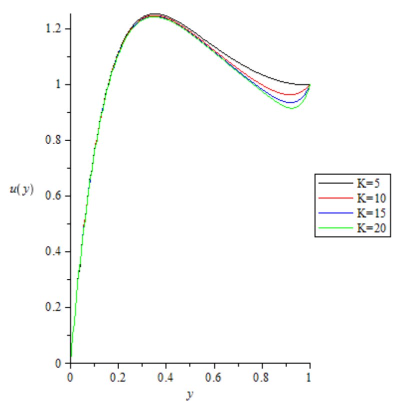
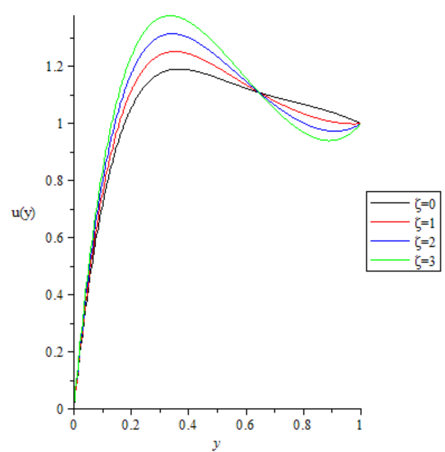


Figure 5: Dimensionless Velocity profile for different Electro-kinetic separation distance based on the plate height K, when $(\zeta = 1, \Lambda_1 = 0.1, \Lambda_2 = 0.2, \Lambda_3 = -2)$



Figuer 6: Dimensionless Velocity profile for different Specific internal Energy ζ when $(K = 5, \Lambda_1 = 0.1, \Lambda_2 = 0.2, \Lambda_3 = -2)$

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